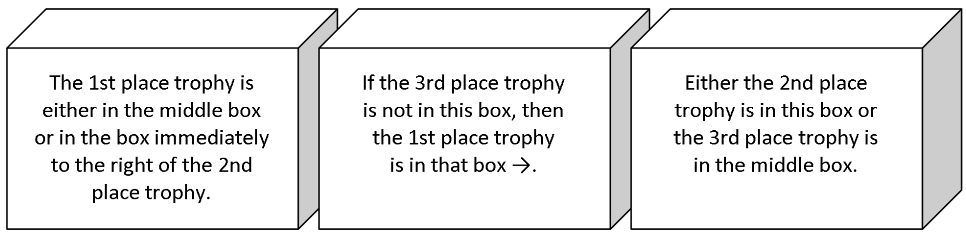
# CONE of PWN

## 2017 CONTEST

### Problems & Solutions

**1.** At the ARML awards ceremony there are three boxes in a row on the awards table. The boxes contain the 1st, 2nd, and 3rd place trophies in some order (there is only one trophy per box). The organizers have "clearly" labeled the boxes from left to right as shown:




To make matters worse, only one of the three labels is correct. What are the contents of the three boxes, from left to right?

**Answer.** $\boxed{\text{Left: 2nd, Middle: 1st, Right: 3rd}}$

**Solution.** Checking the truth value of each of the three labels for the six possible arrangements of trophies gives

| Trophy | | | Label | | |
|---|---|---|---|---|---|
| Left | Middle | Right | Left | Middle | Right |
| 1st | 2nd | 3rd | F | F | F |
| 1st | 3rd | 2nd | F | T | T |
| 2nd | 1st | 3rd | T | F | F |
| 2nd | 3rd | 1st | F | T | T |
| 3rd | 1st | 2nd | T | F | T |
| 3rd | 2nd | 1st | T | T | F |

Since only one label is correct, the arrangement from left to right must be as claimed.

□

**2.** How many positive values of $x$ (in radians) less than 2017 satisfy $\cos^2 x + 2\sin^2 x = 1$?

**Answer.** $\boxed{642}$

**Solution.** Since $\sin^2 x + \cos^2 x = 1$ for all $x$, the given condition is equivalent to $\sin^2 x = 0$, or $\sin x = 0$. So we want the number of positive integer multiples of $\pi$ less than 2017 which is $\lfloor 2017/\pi \rfloor = 642$. Note that some care must be taken when computing $\lfloor 2017/\pi \rfloor$ by hand, since $\lfloor 2017/3.14 \rfloor = 642$ whereas $\lfloor 2017/(22/7) \rfloor = 641$. $\square$

**3.** A list of six positive integers $p, q, r, s, t, u$ satisfies $p < q < r < s < t < u$. There are exactly 15 pairs of numbers that can be formed by choosing two different numbers from the list. The 15 sums of these pairs are 25, 30, 38, 41, 49, 52, 54, 63, 68, 76, 79, 90, 95, 103, 117. What is the value of $r + s$?

**Answer.** $\boxed{54}$

**Solution.** The smallest sum must be the sum of the smallest pair, and similarly for the largest; thus $p + q = 25$ and $t + u = 117$. The sum of all pairs is five times the sum of all fifteen numbers since each of the six values occur in exactly five pairs. Thus

$$5\,(p + q + r + s + t + u) = 25 + 30 + 38 + \cdots + 104 + 117 = 980$$

Substituting $p + q = 25$ and $t + u = 117$, and solving gives $r + s = 54$. $\square$

**4.** A certain type of ID number has six digits, each an integer from 1 to 9 which may be repeated. Furthermore, each of these ID numbers has the following curious property—its first two digits form a number divisible by 2, its first three digits form a number divisible by 3, etc., so that the ID number itself is divisible by 6. One ID number that satisfies these conditions is 123252. How many different possible such ID numbers are there?

**Answer.** $\boxed{324}$

**Solution.** There are 3 possibilities for the first digit given any last 5 digits, because the entire number (and hence, the sum of its digits) must be divisible by 3. Because the first two digits form a number divisible by 2, the second digit must be 2, 4, 6, or 8, which is 4 possibilities. Because the first five digits form a number divisible by 5, the fifth digit must be a 5. Now, if the fourth digit is a 2, then the last digit has two choices (2 or 8), and the third digit has 5 choices (1, 3, 5, 7, or 9). If the fourth digit is a 4, then the last digit must be a 6, and the third digit has 4 choices (2, 4, 6, or 8). If the fourth digit is a 6, then the last digit must be a 4, and the third digit has 5 choices (1, 3, 5, 7, or 9). If the fourth digit is an 8, then the last digit has two choices (2 or 8), and the third digit has 4 choices (2, 4, 6, or 8). So there are a total of

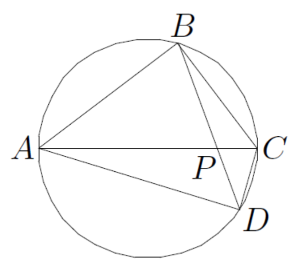
$$3 \cdot 4(2 \cdot 5 + 4 + 5 + 2 \cdot 4) = 3 \cdot 4 \cdot 27 = 324$$

possibilities for the ID number. $\square$

**5.** Quadrilateral $ABCD$ is inscribed in a circle with radius 1 and $AC$ as a diameter, and $|BD| = |AB|$. The diagonals $AC$ and $BD$ intersect at $P$. If $|PC| = 2/5$, how long is side $CD$?

**Answer.** $\boxed{2/3}$

**Solution.**




Let $\angle ACD = 2\theta$. By a routine angle chase, we find $\angle CAD = 90^\circ - 2\theta$, $\angle ABD = 2\theta$, $\angle ADB = 90^\circ - \theta$, and $\angle CDB = \theta$. Applying the Law of Sines to triangle $DCP$ gives

$$\frac{|DP|}{\sin 2\theta} = \frac{2}{5\sin\theta}$$

and to triangle $DAP$ gives

$$\frac{|DP|}{\sin(90^\circ - 2\theta)} = \frac{8}{5\sin(90^\circ - \theta)}$$

Combining these last two equations gives

$$\frac{2\sin 2\theta}{5\sin\theta} = \frac{8\cos 2\theta}{5\cos\theta}.$$

which is successively equivalent to

$$\begin{aligned} 2\sin 2\theta\cos\theta &= 8\cos 2\theta\sin\theta \\ 4\sin\theta\cos^2\theta &= 8\cos 2\theta\sin\theta \\ \cos^2\theta &= 2\cos 2\theta \\ \cos 2\theta + 1 &= 4\cos 2\theta \\ \cos 2\theta &= \frac{1}{3}. \end{aligned}$$

Finally, $|CD| = 2\cos 2\theta = 2/3$. $\square$

**6.** Don and Ken set out to run an ultramarathon from Lehigh to ARMLville at 7:00 A.M. Don ran at a constant rate of 12 mi/hr and Ken ran at 8 mi/hr. After 2 hours Ken realized that he would need to get to ARMLville first (to register the team and organize the dorm keys), so he hopped on the back of a passing train traveling to ARMLville along the same route at a constant rate of 20 mi/hr. Ken soon caught up to Don, and 4 hours after passing Don the train reached ARMLville. What time did Don arrive at ARMLville?

**Answer.** $\boxed{\text{4:40:00 P.M.}}$

**Solution.** After two hours Don has run 24 miles and Ken has run 16 miles, so Ken is 8 miles behind Don at that point. After getting on the train, Ken closes the distance to Don at a rate of $20 - 12 = 8$ mi/hr. So they will meet in one hour, i.e. at 10 A.M., or 36 miles from Lehigh. Ken then takes 4 hours from that point to reach ARMLville, so ARMLville must be $20 \cdot 4 = 80$ miles from the point where the train passed Don. Hence, ARMLville is 116 miles from Lehigh. Running at 12 miles per hour, Don covers this entire distance in 9 hours and 40 minutes, so his finishing time is 4:40 P.M.. $\square$

**7.** For any two positive real numbers $x$ and $y$, the operator $\diamond$ yields a positive real number $x \diamond y$. This operator satisfies the following conditions.

(i) $1 \diamond 1 = 1$

(ii) $(x \diamond 1) \diamond x = x \diamond 1$ for all $x > 0$

(iii) $(x \cdot y) \diamond y = x \cdot (y \diamond y)$ for all $x, y > 0$

Find the value of $20 \diamond 11$.

**Answer.** $\boxed{20}$

**Solution.** For any positive real number $a$,

$$a = a \cdot 1 = a \cdot (1 \diamond 1) = (a \cdot 1) \diamond 1 = a \diamond 1,$$

and

$$a = a \diamond 1 = (a \diamond 1) \diamond a = a \diamond a.$$

So for any positive reals $a$ and $b$

$$a \diamond b = \left(\frac{a}{b} \cdot b\right) \diamond b = \frac{a}{b} \cdot (b \diamond b) = \frac{a}{b} \cdot b = a$$

Thus, $20 \diamond 11 = 20$. $\square$

**8.** If $a$ and $b$ are randomly selected real numbers between 0 and 1, find the probability that the nearest integer to $\dfrac{a-b}{a+b}$ is odd.

**Answer.** $\boxed{1/3}$

**Solution.** We solve this problem geometrically. By symmetry, we may assume $a \geq b$. So we restrict ourselves to the right triangular region

$$R = \{(a, b) \mid 0 \leq b \leq a \leq 1\}.$$

Since $0 \leq \dfrac{a-b}{a+b} \leq 1$, the nearest integer to $\dfrac{a-b}{a+b}$ is odd if and only if

$$\frac{a-b}{a+b} \geq \frac{1}{2},$$

or, equivalently, $a \geq 3b$. The probability we seek is the ratio of the area of the part of this region inside $R$ to the area of $R$. A simple calculation gives the answer. $\square$

**9.** A point $M$ is chosen inside square $ABCD$ such that the measures of both $\angle MAC$ and $\angle MCD$ are $17^\circ$. Find the measure of $\angle ABM$ in degrees.

**Answer.** $\boxed{\text{56 degrees}}$

**Solution.** Let $\omega$ be the circumcircle of triangle $\triangle AMC$. Let $D'$ be a point on the tangent line to $\omega$ at $C$ and on the same side of the line containing the diameter through $C$ as $M$. Arc $MC$ is intercepted by both $\angle MAC$ and $\angle MCD'$, so those two angles have equal measure. Thus $\angle MCD' = \angle MCD$ and so line $CD$ is tangent to $\omega$ at $C$. Since the diameter through $C$ is perpendicular to the tangent line, circumcenter of $\triangle AMC$ is on line $BC$.

Now $AC$ is the diagonal of a square, so $\angle DAC = \angle DCA = 45^\circ$. Thus $\angle MAD = \angle MCA = 45^\circ - 17^\circ = 28^\circ$. Interchanging the roles of $A$ and $C$ in the previous argument shows that the circumcenter of $\triangle AMC$ is on line $AB$ also. Hence $B$ is the circumcenter of $\triangle AMC$. Thus $\angle MBC$ is a central angle of $\omega$ that intercepts the same arc as inscribed angle $\angle MAC$, and hence has twice its measure, or $34^\circ$. The desired angle is the complement of this, so it measures $90^\circ - 34^\circ = 56^\circ$. $\square$

**10.** In how many ways can the set $\{1, 2, \ldots, 22\}$ be partitioned into three nonempty sets so that none of these sets contains a pair of consecutive integers?

**Answer.** $\boxed{1048575}$

**Solution.** We first construct the three subsets disregarding the fact that they must be nonempty. The numbers 1 and 2 must belong to two different subsets. We then have two choices for each of the numbers $3, 4, \ldots, 22$, and different choices lead to different partitions. Hence there are $2^{20}$ such partitions, one of which has an empty part. Thus the final answer is $2^{20} - 1 = 1048575$. $\square$

**11.** How many ordered pairs $(A, B)$ of nonempty sets are there such that $A \cup B = \{1, 2, 3, 4, 5\}$?

**Answer.** $\boxed{241}$

**Solution.** Each of the numbers $1, 2, 3, 4, 5$ must be contained in either $A$ only, $B$ only, or both. There are $3^5$ such possibilities, but two of them have either $A$ empty or $B$ empty. Hence, the answer is $3^5 - 2 = 241$. $\square$

**12.** A rectangle can be divided into $n$ congruent squares. The same rectangle can also be divided into $n + 76$ congruent squares. Find $n$.

**Answer.** $\boxed{324}$

**Solution.** Let $ab = n$ and $cd = n + 76$, where $a, b$ and $c, d$ are the numbers of squares in each direction for the partitioning of the rectangle into $n$ and $n + 76$ squares, respectively. Then $\frac{a}{c} = \frac{b}{d}$, or $ad = bc$. Let $u = \gcd(a, c)$ and $v = \gcd(b, d)$, then there exist positive integers $x$ and $y$ such that $\gcd(x, y) = 1$, $a = ux$, $c = uy$, and $b = vx$, $d = vy$ . Hence we have

$$cd - ab = uv(y^2 - x^2) = uv(y - x)(y + x) = 76 = 2^2 \cdot 19.$$

Since $y - x$ and $y + x$ are positive integers of the same parity and $\gcd(x, y) = 1$, we have $y - x = 1$ and $y + x = 19$ as the only possibility, yielding $y = 10$, $x = 9$ and $uv = 4$. Finally, we have $n = ab = x^2uv = 324$. $\square$

**13.** Two circles, both with the same radius $r$, are placed in the plane without intersecting each other. A line in the plane intersects the first circle at the points $A$, $B$ and the other at the points $C$, $D$ so that

$$|AB| = |BC| = |CD| = 14.$$

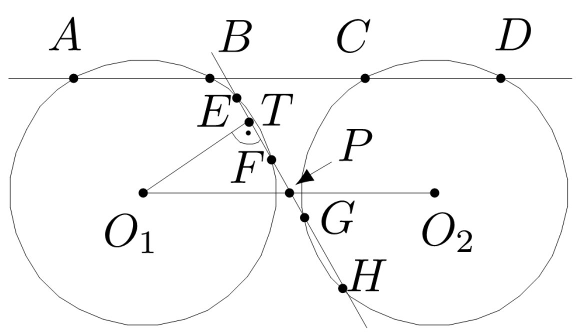
Another line intersects the circles at points $E$, $F$ and $G$, $H$ respectively, so that

$$|EF| = |FG| = |GH| = 6.$$

Find $r$.

**Answer.** $\boxed{13}$

**Solution.** First, note that the centers $O_1$ and $O_2$ of the two circles lie on different sides of the line $EH$, otherwise we have $r < 12$ and $AB$ cannot be equal to 14. Let $P$ be the intersection point of $EH$ and $O_1O_2$.




Points $A$ and $D$ lie on the same side of the line $O_1O_2$, otherwise the three lines $AD$, $EH$, and $O_1O_2$ would intersect in $P$ and $|AB| = |BC| = |CD|$, $|EF| = |FG| = |GH|$ would imply $|BC| = |FG|$, a contradiction. It is easy to see that $|O_1O_2| = 2|O_1P| = |AC| = 28$. Let $h = |O_1T|$ be the height of triangle $O_1EP$.Then we have $h^2 = 14^2 - 6^2 = 160$ from triangle $O_1TP$ and $r^2 = h^2 + 3^2 = 169$ from triangle $O_1TF$. Thus $r = 13$. $\square$

**14.** Find the largest real number $k$ such that

$$\log_{\frac{a}{b}} 2017 + \log_{\frac{b}{c}} 2017 + \log_{\frac{c}{d}} 2017 \geq k \log_{\frac{a}{d}} 2017$$

holds for any positive real numbers $a > b > c > d$.

**Answer.** $\boxed{9}$

**Solution.** Applying a change of base, the given inequality reduces to

$$\frac{\log 2017}{\log \frac{a}{b}} + \frac{\log 2017}{\log \frac{b}{c}} + \frac{\log 2017}{\log \frac{c}{d}} \geq k \cdot \frac{\log 2017}{\log \frac{a}{d}}$$
$$\frac{1}{\log \frac{a}{b}} + \frac{1}{\log \frac{b}{c}} + \frac{1}{\log \frac{c}{d}} \geq k \cdot \frac{1}{\log \frac{a}{d}} \qquad (*)$$

Put $x = \log \frac{a}{b}$, $y = \log \frac{b}{c}$, and $z = \log \frac{c}{d}$. Then since

$$\log \frac{a}{d} = \log \left( \frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{d} \right) = \log \frac{a}{b} + \log \frac{b}{c} + \log \frac{c}{d} = x + y + z,$$

we can rewrite $(*)$ as

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq k \cdot \frac{1}{x + y + z}.$$

By the AM-HM Inequality, we know that $k = 9$ is the largest value for which this holds for all $x, y$, and $z$. $\square$

**15.** Twelve cards lie in a row. The cards are of three kinds—with both sides white (WW), both sides black (BB), or with a white and a black side (WB). Initially, nine of the twelve cards have a black side up. Then cards 1 – 6 are turned over, and subsequently four of the twelve cards have a black side up. Now cards 4 – 9 are turned over, and six cards have a black side up. Finally, the cards 1 – 3 and 10 – 12 are turned over, after which five cards have a black side up. How many cards of each kind were there? Give your answer as a triple of the form (WW, BB, WB).

**Answer.** $\boxed{\text{(WW, BB, WB) = (3, 0, 9)}}$

**Solution.** Denote by $a_1, a_2, \ldots, a_{12}$ the sides of each card that are initially visible, and by $b_1, b_2, \ldots, b_{12}$ the initially invisible sides—each of these is either white or black. The conditions of the problem imply the following:

(a) there are 9 black and 3 white sides among

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12};$$

(b) there are 4 black and 8 white sides among

$$b_1, b_2, b_3, b_4, b_5, b_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12};$$

(c) there are 6 black and 6 white sides among

$$b_1, b_2, b_3, a_4, a_5, a_6, b_7, b_8, b_9, a_{10}, a_{11}, a_{12};$$

(d) there are 5 black and 7 white sides among

$$a_1, a_2, a_3, a_4, a_5, a_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}.$$

Cases (b) and (d) together enumerate each of the sides $a_i$ and $b_i$ exactly once; hence, there are 9 black and 15 white sides altogether. Therefore, all existing black sides are enumerated in (a), implying that we have 9 cards with one black and one white side, and the remaining 3 cards have both sides white. □

**16.** At a tennis tournament, there were twice as many girls participating than boys. Each pair of players had only one match and there were no draws. The ratio of wins by girls to wins by boys was 7/5. What is the least number of players who took part in the tournament?

**Answer.** $\boxed{9}$

**Solution.** Let $n$ be the number of boys, so $2n$ is the number of girls, and $3n$ is the total number of players in the tournament. There are $2n^2$ matches between girls and boys. Let $x$ be the number of those that were won by girls. Thus the number of games won by girls is $\binom{2n}{2} + x$ and the number won by boys is $\binom{n}{2} + 2n^2 - x$. Thus,

$$\frac{\binom{2n}{2} + x}{\binom{n}{2} + 2n^2 - x} = \frac{7}{5}.$$

Solving this equation for $x$ gives

$$x = \frac{n(5n+1)}{8}$$

The smallest value of $n$ for which this is an integer is $n = 3$, and $n = 3, x = 6$ satisfies the conditions of the problem. In this situation 9 players participated in the tournament. $\square$

**17.** Given triangle $ABC$, line $\ell$ is the bisector of the external angle at $C$. The line through the midpoint $O$ of $AB$ parallel to $\ell$ meets $AC$ at $E$. Determine $|CE|$ if $|AC| = 7$ and $|CB| = 4$.

**Answer.** $\boxed{11/2}$

**Solution.** Let $F$ be the intersection point of $\ell$ and the line $AB$. Since $|AC| > |BC|$, the point $E$ lies on the segment $AC$, and $F$ lies on the ray $AB$. Let the line through $B$ parallel to $AC$ meet $CF$ at $G$. Then the triangles $AFC$ and $BFG$ are similar. Moreover, we have $\angle BGC = \angle BCG$, and hence the triangle $CBG$ is isosceles with $|BC| = |BG|$. Hence

$$\frac{|FA|}{|FB|} = \frac{|AC|}{|BG|} = \frac{|AC|}{|BC|} = \frac{7}{4}.$$

Therefore

$$\frac{|AO|}{|AF|} = \frac{\frac{3}{2}}{7} = \frac{3}{14}.$$

Since the triangles $ACF$ and $AEO$ are similar,

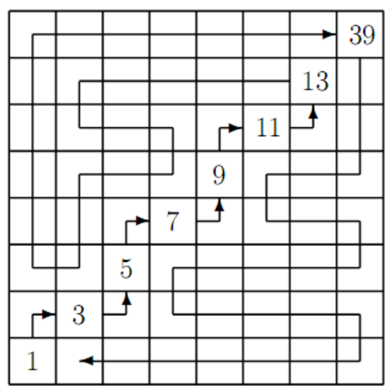
$$\frac{|AE|}{|AC|} = \frac{|AO|}{|AF|} = \frac{3}{14},$$

whence $|AE| = 3/2$ and $|EC| = 11/2$. $\square$

**18.** The squares of an 8 by 8 chessboard are to be labeled with the numbers $1, 2, \ldots, 64$ in such a way that any pair of consecutive integers must label squares that share a common edge (vertically or horizontally). What is the minimum possible sum of the eight labels on one of the main diagonals?

**Answer.** $\boxed{88}$

**Solution.** Since consecutive numbers occupy sqaures of opposite colors, we may assume that all numbers on black squares are odd and all numbers on white squares are even. The diagram below shows that the sum may be as small as 1+3+5+7+9+11+13+39=88.




Suppose it is possible to improve on this. Clearly, the diagonal in question should contain odd numbers, and the largest would have to be at most 37. Once this number is put down, we must remain on the same side of this diagonal. There are exactly 16 black squares and 12 white squares on each side. Hence that largest number is 37 and only one square on the largely empty side of the diagonal has been filled. However, there are 13 odd numbers from 38 to 64 but we have at most 12 white squares to accommodate them. Hence improvement over 88 is impossible. □

**19.** Find $A/B$ if

$$A = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{2017\cdot 2018}$$

and

$$B = \frac{1}{1010\cdot 2018} + \frac{1}{1011\cdot 2017} + \cdots + \frac{1}{2018\cdot 1010}.$$

**Answer.** $\boxed{1514}$

**Solution.** By arithmetic,

$$\begin{aligned}
A &= \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{2017\cdot 2018}\\
&= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2017} - \frac{1}{2018}\\
&= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2017} + \frac{1}{2018} - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2018}\right)\\
&= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2017} + \frac{1}{2018} - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{1009}\right)\\
&= \frac{1}{1010} + \frac{1}{1011} + \frac{1}{1012} + \cdots + \frac{1}{2018}
\end{aligned}$$

Consequently,

$$\begin{aligned}
2A &= \left(\frac{1}{1010} + \frac{1}{2018}\right) + \left(\frac{1}{1011} + \frac{1}{2017}\right) + \cdots + \left(\frac{1}{2018} + \frac{1}{1010}\right)\\
&= 3028\left(\frac{1}{1010\cdot 2018} + \frac{1}{1011\cdot 2017} + \cdots + \frac{1}{2018\cdot 1010}\right)\\
&= 3028B
\end{aligned}$$

Thus $A/B = 1514$. $\square$

**20.** All integers greater than one but less than 100 are put into a hat and two of them are secretpely drawn. Sammy is only given their sum and Puddy is only given their product.

**Sammy**: "I can see that you don't know the two numbers."

**Puddy**: "Now I do know them."

**Sammy**: "Now I do too!"

What are the two numbers?

**Answer.** $\boxed{\text{4 and 13}}$

**Solution.** Let $x$ and $y$ denote the two numbers, and let $S$ and $P$ denote their sum and product. Because initially Puddy cannot deduce the values of $\{x, y\}$, we know that

(i) $P$ cannot be the product of two primes, and

(ii) $P$ cannot be the product of a prime greater than 50 and some integer greater than 1.

Consequently, because Sammy knows of Puddy's initial ignorance of $\{x, y\}$, we find that

(i) $S$ cannot be the sum of two primes, and

(ii) $S$ cannot be the sum of a prime greater than 50 and some integer greater than 1.

This eliminates a lot of possibilities. By (ii), $S$ cannot be greater than 54. By (i), $S$ cannot be even because Goldbach's Conjecture (which states every even number is the sum of two primes) is known to hold for even numbers much greater than 54. Also by (i), $S$ cannot be sum of an odd prime and the prime number 2. So the remaining possibilities for $S$ are those integers not exceeding 53 which are the sum of an odd composite and 2, namely,

$$11, 17, 23, 27, 29, 35, 37, 41, 47, 51, 53. \tag{$*$}$$

We can rule out all but one of these possibilities.

Now after Sammy's first comment, Puddy knows of the limited possibilities for $S$. From the fact that Puddy then knows the values of $\{x, y\}$ we deduce that there is only one way of factoring $P$ into two factors that sum to one of the values in $(*)$. For instance, if $P = 18$, then Puddy realizes that $\{x, y\}$ is either $\{2, 9\}$ or $\{3, 6\}$; but $\{3, 6\}$ is out of the question since their sum is not one of $(*)$. Hence, Puddy deduces that $\{x, y\}$ is in fact $\{2, 9\}$. As another example, if $P = 24$, then Puddy can use similar reasoning to deduce that $\{x, y\}$ is $\{3, 8\}$ (both $\{2, 12\}$ and $\{4, 6\}$ are out of the question since their sum if not one of $(*)$).

Now in these two examples just given, we have $S = 11$. This eliminates $S = 11$ as a possibility now! After Puddy's realization of the two numbers, Sammy supposedly also deduces the values of $\{x, y\}$. However, in these examples, Sammy knows $S = 11$, but he wouldn't know if $P = 18$ or $P = 24$, that is, $\{x, y\}$ is $\{2, 9\}$ or $\{3, 8\}$, or possibly something else.

Completing the proof now is now routine. A similar argument eliminates all other possibilities in $(*)$ but 17. With 17, there is only one pair of summands, namely $\{4, 13\}$, whose product factors in only one way such that the sum of those factors is listed in $(*)$. We leave this to the reader to verify. $\square$