PROVE IT! MATH ACADEMY

2017 Admissions Test

Can you *Prove it!*?

Instructions for the Student: *Complete as many of the parts of the following four problems as you can. Show your work and fully explain all of your answers. You may either type your solutions and print them to a pdf file, or write them by hand and scan them to a pdf file. Submit your solutions by uploading that pdf file along with your application online at* `proveitmath.org`*.*

Some of these problems are intentionally challenging and you are not expected to solve them all completely. They are designed to give you the chance to demonstrate how you wrestle with mathematical problems. Partial solutions and observations should be submitted in the event that you do not find a complete solution. All work should be entirely your own. You should not discuss the problems with anyone else, whether online or in person. Have fun!

1. Three prairie dogs, Alpha, Beta, and Gamma, have peculiar language habits. One of them always tells the truth, another always lies, and the third tells the truth if and only if the previously spoken statement it heard was true.

   (a) The three prairie dogs have the following conversation.

   Alpha: "Gamma always lies."
   Beta: "I never lie."
   Gamma: "Yes you do!"
   Alpha: "No, he doesn't!"

   Which is the liar, which is the truth teller, and which is the copycat?

   (b) The prairie dogs' friend Delta was hanging around during their conversation in part (a). Delta also lies or tells the truth based only on the truth values of the sequence of previous statements he has heard. He decides to pipe up next to continue the conversation:

   Delta: "Sometimes I lie when I speak just after Beta."
   Beta: "Delta, if you are the next one to speak, you will certainly lie."
   Gamma: "Delta, if you are the next one to speak, you will certainly lie."
   Alpha: "If Delta would lie if he were to speak next, then sometimes Delta needs to consider more than the previous two statements he has heard when deciding whether or not to lie."

   How does Delta decide when to tell the truth?

2. At Glacier Ice Cream, there are n flavors of ice cream to pick from and n different toppings. The way you order a sundae is as follows: you pay m dollars which gets you a total of exactly m items, where an item can be a scoop of ice cream or a topping. You may order multiple scoops of a single flavor but you may not order a topping more than once (however, you can order toppings with no scoops). The arrangement of the scoops and toppings on the plate is not noticed, only how many of which flavor and which toppings appear.

   (a) If there are five flavors of ice cream and five toppings available, how many ways can you order a sundae that costs \$6 and has an even number of toppings? An odd number of toppings?

(b) Prove that for all positive natural numbers n and m, there are exactly as many m-item sundaes with an even number of toppings as there are m-item sundaes with an odd number of toppings.

3. Three thin, strong clotheslines are stretched tight across a wide, long alley. You have a spherical ball of radius 1 foot and you are trying to determine if you can place the ball on the clotheslines without it rolling around or falling off.

Note: Gravity pulls objects down towards the ground. We use the term "horizontal" to mean any direction parallel to the ground, and "vertical" to mean perpendicular to the ground. The walls to which the clotheslines are attached are parallel to each other, vertical, and run north-south.

(a) Suppose the clotheslines are all horizontal at the same height, and crisscross each other so as to form a triangle ABC in the middle of the alley. The triangle is small enough that the ball can be placed on top of it without falling through. What are all possible values for the area of triangle ABC? What about for the area of triangle PQR, where P, Q, and R are the points at which the ball touches the three clotheslines?

(b) Suppose the clotheslines are placed as in part (a), such that $|AB| = 1.5$ feet, $|BC| = 2$ feet, and $|CA| = 2.5$ feet. How far does the ball sink below the level of the clotheslines? In other words, what is the distance from the lowest point of the ball to the plane ABC?

(c) Before you went to bed, you decided to place the three clotheslines across the alley so that they were all stretched east-west in the same horizontal plane, each a distance of 0.5 feet from the next. The next morning, you noticed that someone had adjusted your setup! They had tilted the middle clothesline vertically 45° (by raising the western end), and the two outer clotheslines were tilted vertically 45° in the other direction (by raising their eastern ends). You notice the ball can be balanced in a stable position on these clotheslines. If the three tangent points to the clotheslines in this position are P, Q, and R, find the area of triangle PQR.

4. Harry and Hermione are playing with magical rocks, paper, and scissors — whenever any two of the three touch, they change into the third (i.e., if a paper covers a rock they change into a single scissors, if a scissors cuts a paper they become a single rock, and if a rock breaks a scissors they become a sheet of paper). To begin, Harry places some rocks in one pile and Hermione places some papers in another pile. They then take turns combining all of the objects in one pile with an equal number of items from the other pile, placing the resulting items in a new pile. This continues until items can no longer be combined.

(a) How many items and which type of item will remain at the end of the game if they start with thirty rocks and eighteen papers? Four rocks and fourteen papers? Three rocks and ten papers?

(b) If they begin with m rocks and n papers, how many items are left at the end of the game in terms of m and n?

(c) In part (b), which type remains?